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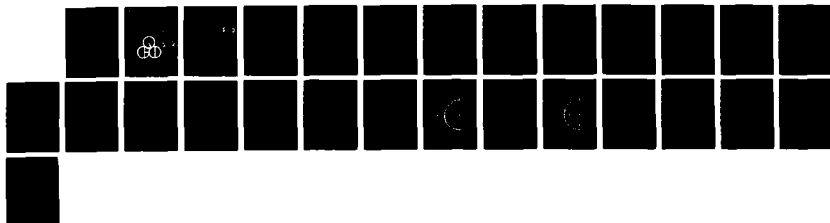
VISIBILITY PROBABILITIES ON LINE SEGMENTS IN THREE  
DIMENSIONAL SPACES SUB. (U) STATE UNIV OF NEW YORK AT  
BINGHAMTON CENTER FOR STATISTICS QU. N YADIN ET AL.  
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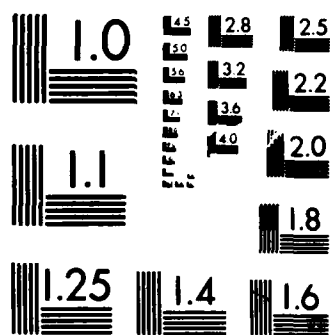
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Visibility Probabilities On Line Segments  
In Three Dimensional Spaces Subject  
To Random Poisson Fields Of Obscuring Spheres.

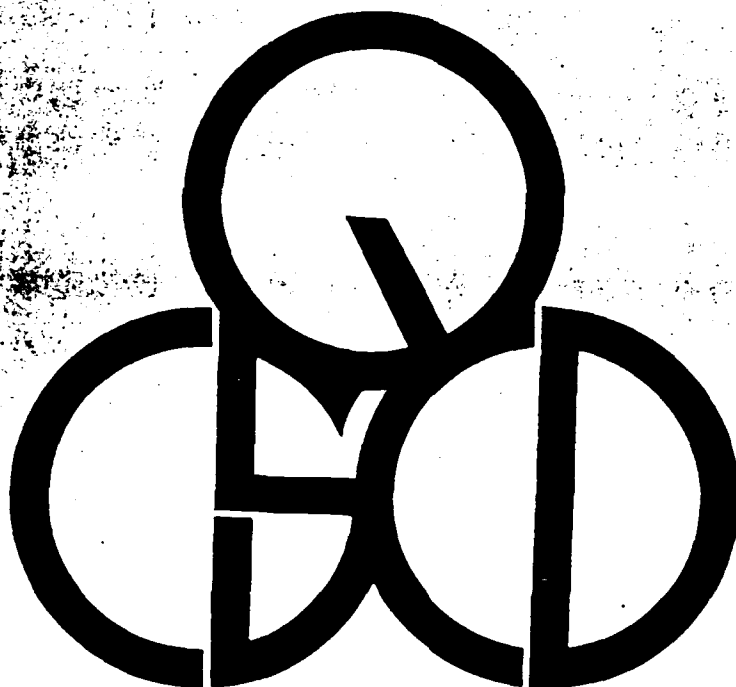
by

M. Yadin and S. Zacks

Technical Report No. 3

March 1, 1987

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Prepared Under Contract DAAG29-84-K-0191  
with the U.S. Army Research Office

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Center For Statistics, Quality Control And Design,  
University Center of the State University  
of New York at Binghamton

Visibility Probabilities On Line Segments  
In Three Dimensional Spaces Subject  
To Random Poisson Fields Of Obscuring Spheres.\*

by

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Technion, Israel Institute of Technology  
and State University of New York at Binghamton

*Abstract*

The methodology of determining simultaneous visibility probabilities and points on line segments is extended to problems of three dimensional spaces, with Poisson random fields of obscuring spheres. Required functions are derived analytically and a numerical example is given for a special case of a standard Poisson field, with uniform distribution of sphere diameters.



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**Key Words:** Poisson Shadowing Processes, Lines of Sight; Visibility Probabilities; Measures of Visibility; Moments of Visibility.

\*Research supported by contract DAAG29-84-K-0191 with the U.S. Army Research Office.

## 1. Introduction

Consider a source of light located at a point  $Q$  and an arbitrary straight line segment,  $C$ , in a three dimensional Euclidean space,  $R^3$ . A layer of random spheres is located between  $Q$  and  $C$ . The centers of the spheres are randomly dispersed between two parallel planes, according to a Poisson process with a specified intensity. Furthermore, the radii of the spheres are independent identically distributed random variables, having a specified distribution, such that both  $Q$ , and the line segment,  $C$ , are not covered by the random spheres. A point  $P$  on  $C$  is called visible (in the light), if the line segment  $\overline{QP}$  does not intersect any random sphere. The measure of visibility on  $C$  is the total portion of  $C$  which is visible. We are interested in the distribution of this random variable.

In a previous study [2, 3, 4] we developed the methodology for determining the moments of the measure of visibility on star-shaped curves in the plane and provided also an approximation to its distribution. The objective of the present paper is to apply the two-dimensional methodology for solving the three dimensional problem mentioned above. The results of the present study are thus restricted to the case where the target curve  $C$ , in the three dimensional space, is a straight line rather than an arbitrary curve. A runway in an airfield as  $C$ , and the pilot's position in a flying airplane as  $Q$ , is a relevant example. In the case under consideration, one can consider the shadowing process on the plane containing  $C$  and  $Q$ . In a previous Technical Report [5] we presented the basic ideas, but evaluated certain key functions by two dimensional numerical integration. This type of numerical solution is time consuming and contributing to numerical error. In the present paper we provide an analytical solution to the problem of determining simultaneous visibility probabilities of arbitrary points on  $C$ . With this analytical solution one can compute moments of random visibility measures on  $C$  very fast and accurately. Moreover, discrete approximations to the distributions of lengths of shadows on  $C$  (see Yadin and Zacks [6]) can now be computed efficiently.

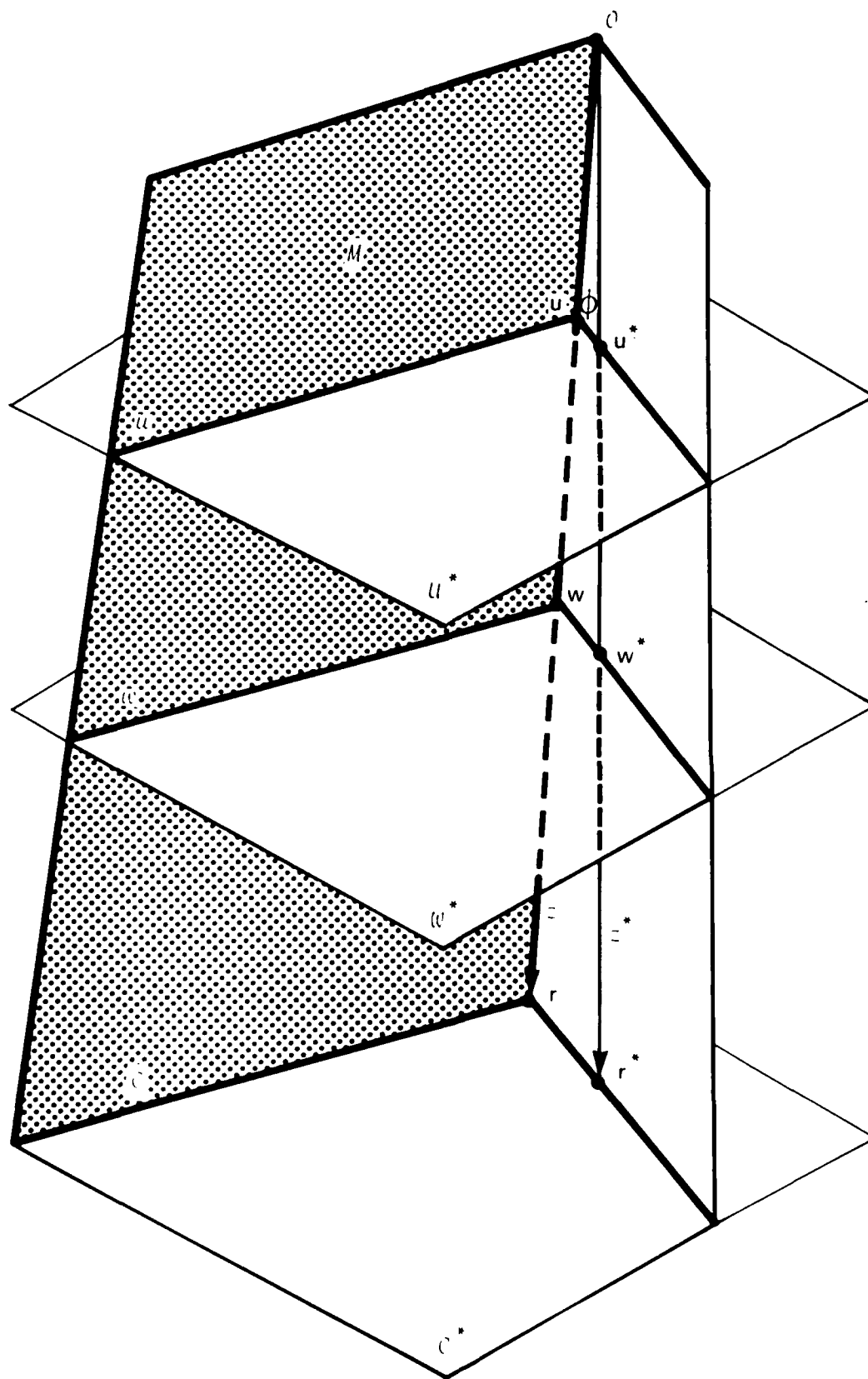


Fig. 1. The Geometry of the Three Dimensional Model.

In Section 2 we present the three dimensional shadowing model, and the basic geometry. In Section 3 we discuss the reduction to a two-dimensional shadowing process. In Section 4 we present the formulae for computing simultaneous visibility probabilities on a segment of  $C$ . Analytical development of some of the functions specified in Section 4 is given in the Appendix. In Section 5 we provide an explicit solution for a special case, in which the diameters of shadowing spheres, in the Poisson field of obscuring elements, are independent and having an identical uniform distribution on  $[0, b]$ .

The results of the present study can be applied to a large class of military and civil problems of visibility in a three dimensional scenario, with randomly scattered obscuring elements.

## 2. The Three Dimensional Shadowing Model.

Consider a source of light located at a point  $Q$  (the origin), and an arbitrary straight line,  $C$ , in a three-dimensional Euclidean space. A layer of random spheres is located between two planes  $U^*$ ,  $W^*$ , which are parallel to the line  $C$ . Let  $C^*$  be a plane containing  $C$ , parallel to  $U^*$  (see Figure 1). The random spheres, whose centers are located between  $U^*$  and  $W^*$  cast shadows on  $C$ . As stated in the Introduction, the objective is to develop formulae for the computation of the probabilities of simultaneous visibility of any  $n$  points on  $C$  ( $n \geq 1$ ). These probabilities are then applied to determine the moments of total visibility measures on  $C$ , and to approximate its distribution. In the present section we discuss the basic stochastic and geometric structure of the field.

Let  $u^*$ ,  $w^*$  and  $r^*$  be the distances of  $U^*$ ,  $W^*$  and  $C^*$  from  $Q$ , respectively. It is assumed that

$$(2.1) \quad 0 < b \leq u^* < w^* < r^* - b,$$

where  $b$  is the maximal radius of a sphere. Let  $M$  be a plane passing through  $Q$  and  $C$ . Let  $U$  and  $W$  be straight lines at which  $M$  intersects  $U^*$  and  $W^*$ . The three parallel



lines,  $\mathcal{U}$ ,  $\mathcal{W}$  and  $\mathcal{C}$  are at distances  $u$ ,  $w$  and  $r$  from  $\mathcal{Q}$ . The following relationship is satisfied:

$$(2.2) \quad \frac{u^*}{u} = \frac{w^*}{w} = \frac{r^*}{r} = \cos(\phi),$$

where  $\phi$  is the angle between the axes  $Z^*$  and  $Z$ , which pass through  $\mathcal{Q}$  and are perpendicular to  $\mathcal{C}^*$  and  $\mathcal{C}$ , respectively. The spheres which are scattered between  $\mathcal{U}^*$  and  $\mathcal{W}^*$  have random locations and random radii. It is assumed that the centers of spheres follow a random Poisson field of intensity  $\lambda$ , and that the radii of disks are independent random variables  $X_1, X_2, \dots$  having a common c.d.f.  $F(x)$ , concentrated on  $[0, b]$ . More specifically, let  $(c_1, c_2, c_3, x)$  be the three location coordinates of the center of a sphere and its radius. Let  $S$  be the sample space of randomly scattered spheres, and let  $\mathcal{B}$  denote the Borel  $\sigma$ -field of subsets of  $S$ . For  $B \in \mathcal{B}$ , let  $N\{B\}$  be the number of random spheres having coordinates in  $B$ . It is assumed that for each  $m \geq 2$ , and every partition  $\{B_1, \dots, B_m\}$  of  $S$ ,  $N\{B_1\}, \dots, N\{B_m\}$  are independent random variables, having Poisson distributions with means

$$(2.3) \quad \Lambda(B_i) = \lambda \int \dots \int_{B_i} dF(x) dc_1 \dots dc_3, \quad i = 1, \dots, m.$$

The random spheres in the Poisson field cast shadows on  $\mathcal{C}^*$ , and in particular on the line  $\mathcal{C}$ . In order to study the shadowing process on  $\mathcal{C}$  we consider in the next section the projection of the Poisson field on the plane  $\mathcal{M}$ .

### 3. The Two-Dimensional Shadowing Model on $\mathcal{M}$ .

In the present section we consider the reduction of the shadowing model to  $\mathcal{M}$ . In particular, we have to determine the distribution of the centers of shadowing disks on  $\mathcal{M}$  and the conditional distribution of their radii given the location of their centers. In Figure 2 we present the geometry of the reduction to a two-dimensional model by the projection of the three-dimensional space on a plane perpendicular to  $\mathcal{C}$ . We restrict attention to spheres

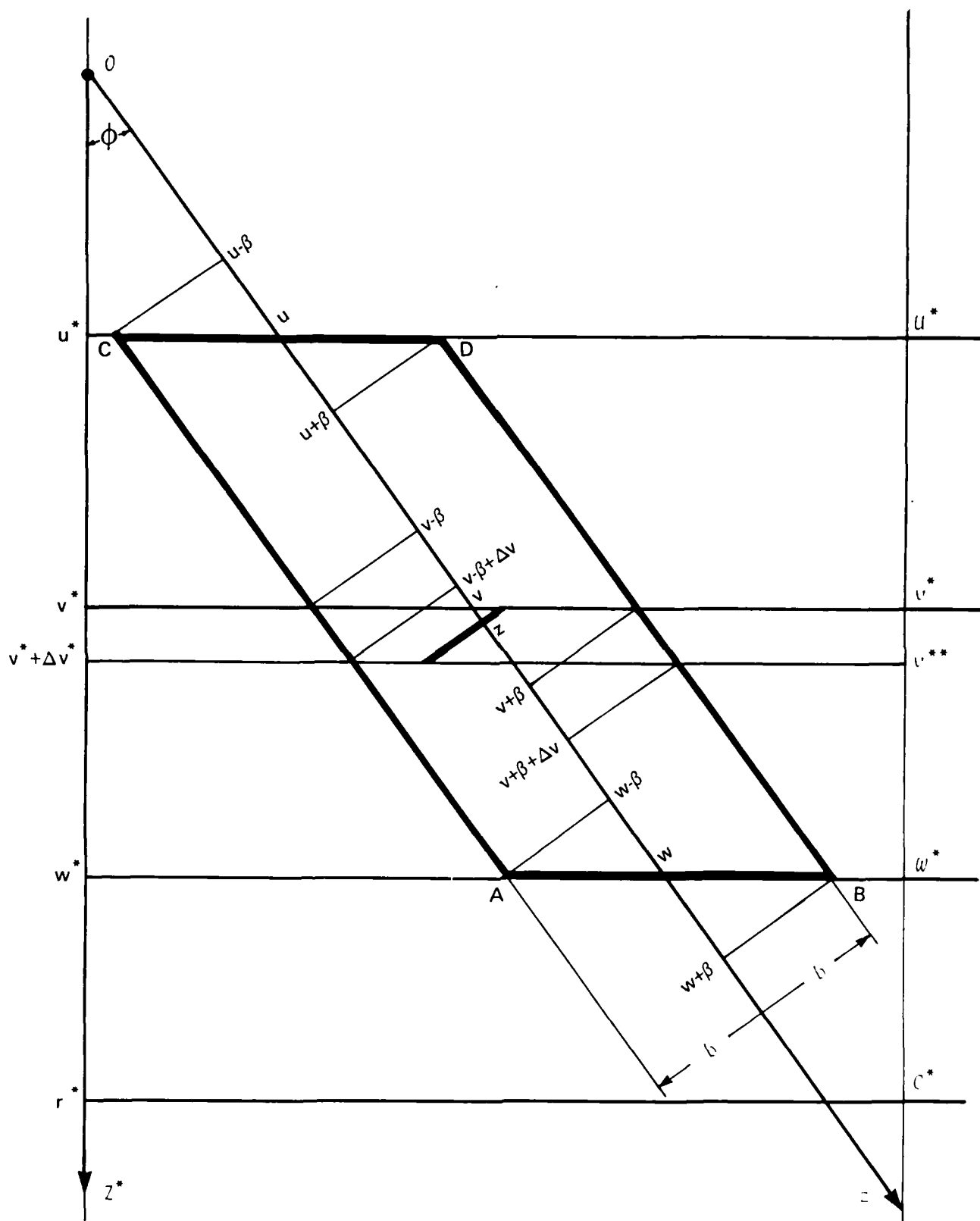


Fig. 2. The Geometry of the Reduction to A Two Dimensional Model

which are centered within a prism,  $P$ , whose projection is the parallelogram  $ABCD$  (see Figure 2). The distances from  $M$  of spheres centered outside this prism is greater than  $b$ . Accordingly, such spheres do not intersect  $M$  and cannot cast shadows on  $C$ . A sphere centered within the prism  $P$  intersects  $M$  if its radius,  $x$ , is greater than its distance  $d$  from  $M$ . The intersection of such a sphere with  $M$  is a disk of radius  $y = (x^2 - d^2)_+^{1/2}$ , where  $a_+ = \max(a, 0)$ . For simplicity, we will assume that all the spheres centered within the prism  $P$  generate on  $M$  disks; those spheres with  $x \leq d$  generate on  $M$  a disk with radius  $y = 0$ . The center of a disk generated by a sphere is the orthogonal projection of its center on  $M$ . Such projections may lie between two parallel lines on  $M$ , having distances  $u - \beta$  and  $w + \beta$  from  $\underline{Q}$ , where

$$(3.1) \quad \beta = b \tan \phi.$$

The orthogonal projection of centers of random spheres generate on  $M$  a two-dimensional Poisson process. The intensity of this process at any given point is proportional to the length of the segments within the prism  $P$  on the normal to  $M$  through the point. As seen in Figure 2, this intensity decreases to zero towards the edges. In the special case of  $\beta = 0$  the Poisson process on  $M$  is homogeneous with intensity  $\eta = 2\lambda b$ . Generally, the intensity of the Poisson process on  $M$ , at all points on a line parallel to  $\underline{U}$  at distance  $z$  from  $\underline{Q}$  is

$$(3.2) \quad \eta(z) = \lambda \frac{b}{\beta} [\min(z + \beta, w) - \max(z - \beta, u)].$$

Notice that if  $u + \beta \leq z \leq w - \beta$  then  $\eta(z) = 2\lambda b$ . That is, the Poisson field on  $M$  is homogeneous only between the parallel lines of distance  $u + \beta$  and  $w - \beta$  from  $\underline{Q}$ .

Consider a sphere centered in the prism  $P$ , on a plane  $\mathcal{V}^*$ , whose distance from  $M$  is  $a$ . This sphere generates on  $M$  a disk whose center falls on a line of distance  $z$  from  $\underline{Q}$  (see Figure 2). Moreover,

$$(3.3) \quad |z - v| = a\beta/b.$$

Accordingly, if  $x$  is the radius of the sphere, the radius of the disk is

$$(3.4) \quad y = ((x^2 - (z - v)^2 b^2 / \beta^2)_+)^{1/2}.$$

Let  $G(y | z, v)$  denote the *conditional CDF* of the random radius  $Y$ , of a disk on  $M$ , given that its center is on a line of distance  $z$  from  $\underline{Q}$ , and the center of the generating disk is on a plane intersecting  $M$  at distance  $v$  from  $\underline{Q}$ . According to (3.4),

$$(3.5) \quad G(y | z, v) = \begin{cases} 0, & y < 0 \\ F((y^2 + (z - v)^2 b^2 / \beta^2)^{1/2}), & 0 \leq y < b(1 - \frac{z - v}{\beta})^2)^{1/2} \\ 1, & b(1 - \frac{z - v}{\beta})^2)^{1/2} \leq y. \end{cases}$$

It is interesting to notice that  $G(C | z, v) = F(\frac{b}{\beta}|z - v|)$ .

In the following section we show how the simultaneous visibility probability  $p(s_1, \dots, s_n)$  can be computed from  $G(y | z, v)$ , and the uniform scattering in the three dimensional prism  $\mathcal{P}$ . For the direct computations we partition the layer between the two parallel planes  $\mathcal{U}^*$  and  $\mathcal{W}^*$  to parallel layers of width  $\Delta^*$  ( $\Delta^* \rightarrow 0$ ). Consider a layer between the parallel planes  $\mathcal{V}^*$  and  $\mathcal{V}^{**}$ , within the original layer, of distances  $v^*$  and  $v^* + \Delta^*$  from  $\underline{Q}$ ;  $u^* < v^* < v^* + \Delta^* < w^*$ . Let  $v + \Delta v$  be the distance from  $\underline{Q}$  at which  $\mathcal{V}^{**}$  intersects  $M$ . We have

$$(3.8) \quad \frac{v^*}{v} = \frac{v^* + \Delta^*}{v + \Delta v} = \frac{r^*}{r} = \cos \phi.$$

Every point in  $M$ , which lies on a line parallel to  $\mathcal{V}$ , whose distance from  $\underline{Q}$  is between  $v - \beta + \Delta v$  to  $v + \beta$  has the same intensity  $\mu(z)\Delta v$ , where

$$(3.9) \quad \mu(z) = \lambda b / \beta,$$

as can be readily verified from Figure 2. Since  $\Delta v \rightarrow 0$ , we can consider the Poisson fields on these thin layers to be almost homogeneous.

#### 4. Simultaneous Visibility of Points on $C$ .

Consider  $n$  points on  $C$ , between  $\underline{P}_{s'}$  and  $\underline{P}_{s''}$ . Let  $s_1 < \dots < s_n$  be the coordinates of these points, with respect to the orthogonal projection of  $\underline{Q}$  on  $C$ . In the present section we derive a formulae for the simultaneous visibility probability,  $p(s_1, \dots, s_n)$ , of these  $n$  points.

We say that a specified set of  $n$  points on  $C$  are simultaneously visible, if the random disks on  $M$  do not intersect any one of the  $n$  rays from  $\underline{Q}$  to  $\underline{P}_{s_i}$  ( $i = 1, \dots, n$ ).

We introduce the points  $\underline{P}_{\theta'}$  and  $\underline{P}_{\theta''}$ , with  $\theta' < s'$  and  $\theta'' > s''$  so that, disks centered on the left of the ray from  $\underline{Q}$  to  $\underline{P}_{\theta'}$ , or on the right of the ray from  $\underline{Q}$  to  $\underline{P}_{\theta''}$ , will not cast shadow on the interval  $\bar{C} = [\underline{P}_{s'}, \underline{P}_{s''}]$ . The coordinates of  $\underline{P}_{\theta'}$  and  $\underline{P}_{\theta''}$  are:

$$(4.1) \quad \theta' = s' - b((s')^2 + r^2)^{1/2} / (u - \beta),$$

and

$$(4.2) \quad \theta'' = s'' + b((s'')^2 + r^2)^{1/2} / (u - \beta).$$

Let  $C$  be the set of points in  $M$  such that, random disks which are centered at  $C$  can cast shadows on the interval  $\bar{C}$ .  $C$  is a trapezoid bounded by the lines parallel to  $C$ , at distances  $u - \beta$  and  $w + \beta$  from  $\underline{Q}$ , and by the line segments  $\overline{\underline{Q}\underline{P}_{\theta'}}$  and  $\overline{\underline{Q}\underline{P}_{\theta''}}$ .

Let  $\mu\{C\}$  denote the expected numbers of disks having centers in  $C$ . We consider also disks of radius  $y = 0$ . According to (3.9)

$$(4.3) \quad \begin{aligned} \mu\{C\} &= \frac{\lambda b}{\beta} \cdot \frac{\theta'' - \theta'}{r} \int_u^w dv \int_{v-\beta}^{v+\beta} z dz \\ &= \frac{\lambda b}{r} (\theta'' - \theta') (w^2 - u^2). \end{aligned}$$

For values of  $s$  in  $(\theta', \theta'')$  and  $0 \leq t < \theta'' - \theta' - s$ , we denote by  $\lambda K_+(s, t)$  [resp.  $\lambda K_-(s, t)$ ] the expected number of disks centered between the line segments  $\overline{\underline{Q}\underline{P}_s}$  and  $\overline{\underline{Q}\underline{P}_{s+t}}$  [resp.  $\overline{\underline{Q}\underline{P}_{s-t}}$ ], which do not intersect  $\overline{\underline{Q}\underline{P}_s}$ .

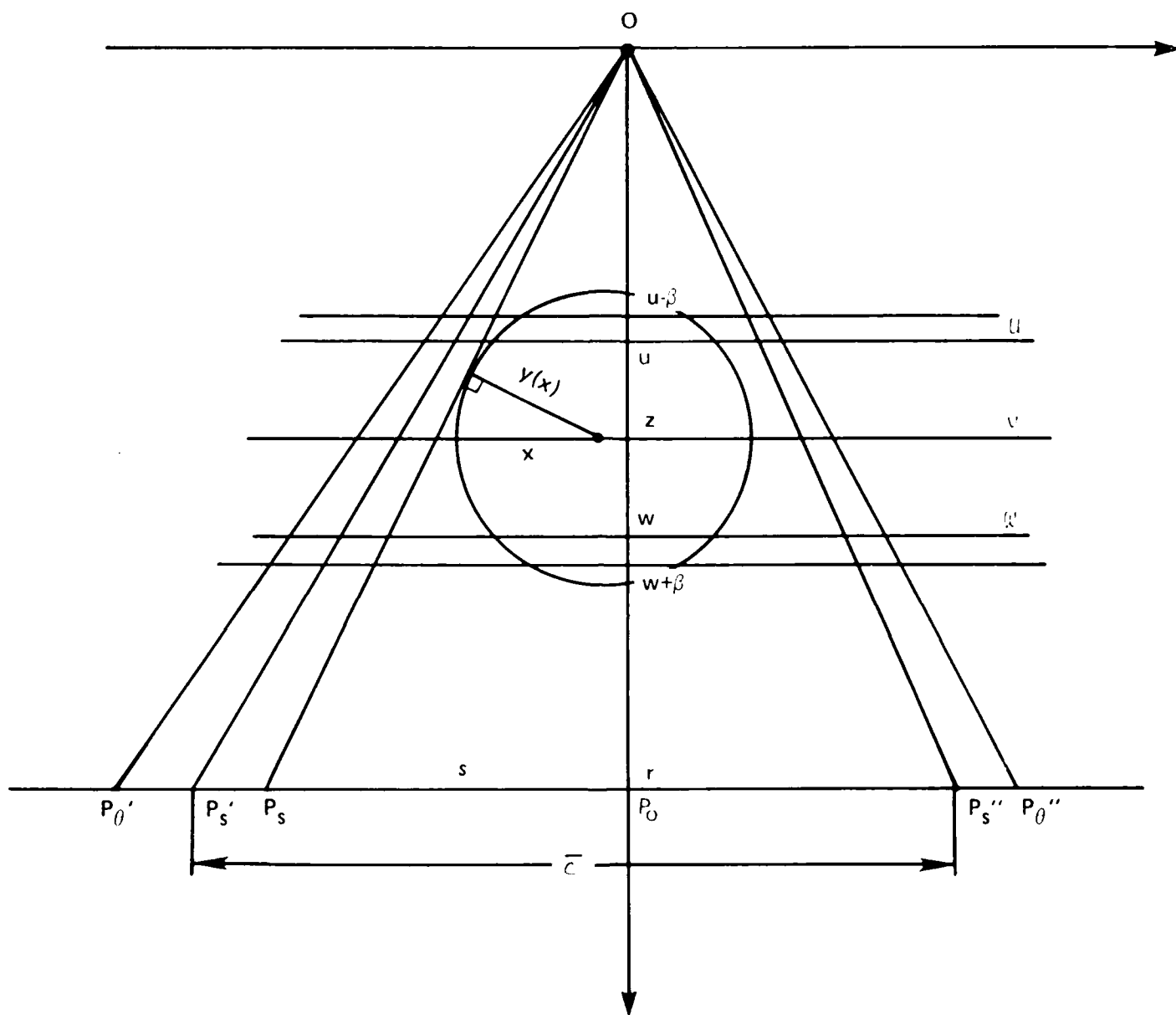


Fig. 3. The Geometry On  $M$ : Non-Standard Case

Consider a disk centered on a line  $\mathcal{V}$ , parallel to  $\mathcal{U}$ , which intersects the axis  $Z$  at distance  $z$  from  $Q$ ,  $u - \beta \leq z \leq w + \beta$ , and having a distance  $x$  from the intersection of  $\mathcal{V}$  from  $\overline{OP_s}$  (see Figure 3). The maximal radius that such a disk can have, without intersecting  $\overline{OP_s}$ , is

$$(4.4) \quad y(x) = xr/(s^2 + r^2)^{1/2}, \quad 0 \leq x \leq \frac{z}{t}.$$

Hence, the function  $K_+(s, t)$  is given by,

$$(4.5) \quad K_+(s, t) = \frac{b}{\beta} \int_u^w dv \int_{v-\beta}^{v+\beta} dz \int_0^{zt/r} G\left(\frac{xr}{(s^2 + r^2)^{1/2}} \mid v, z\right) dx.$$

Geometrical considerations yield that  $K_+(s, t) = K_-(s, t) = K(x, t)$ , for every  $s \in (\theta', \theta'')$  and all  $t \geq 0$ .

In the Appendix we prove that

$$(4.6) \quad K(s, t) = \frac{b\beta^2 t}{c^2 r} \left[ \frac{c^2}{\beta^2} (w^2 - u^2) - \frac{\pi c}{\beta} (w - u) \Psi(0) + Q\left(\frac{c}{\beta} u, c\right) - Q\left(\frac{c}{\beta} w, c\right) \right],$$

where  $c = \alpha\beta t/b$ ,  $\alpha = 1/(s^2 + r^2)^{1/2}$ ,

$$(4.7) \quad \Psi(\tau) = \int_{\tau}^1 [1 - F(b\rho)] \rho d\rho, \quad 0 \leq \tau \leq 1,$$

and  $Q(v, c)$  is given by formula (A.13). In Section 6 we present an explicit formula for  $K(s, t)$ , for the case of a uniform distribution of radii of spheres.

The simultaneous visibility probability  $p(s_1, \dots, s_n)$  can be expressed in terms of the  $K(s, t)$  function, as in Yadin and Zacks [3, 4]. For  $n = 1$ , let  $p(s)$  denote the visibility probability of  $P_s$ . This is given by

$$(4.8) \quad p(s) = \exp\{-[\mu\{C\} - \lambda(K(s, s - \theta') + K(s, \theta'' - s))]\},$$

for every  $\theta' \leq s \leq \theta''$ . For  $n \geq 2$ ,

$$(4.9) \quad p(s_1, \dots, s_n) = \exp\{-[\mu\{C\} - \lambda(K(s_1, s_1 - \theta') + K(s_n, \theta'' - s_n)) - \lambda \sum_{i=1}^{n-1} (K(s_i, \tilde{s}_i - s_i) + K(s_{i+1}, s_{i+1} - \tilde{s}_i))]\},$$

where

$$(4.10) \quad \tilde{s}_i = r \tan \left( \frac{1}{2} \left( \tan^{-1} \left( \frac{s_i}{r} \right) + \tan^{-1} \left( \frac{s_{i+1}}{r} \right) \right) \right).$$

Furthermore, the probability that the whole interval  $\bar{C} = [P_{s'}, P_{s''}]$  is visible is

$$(4.11) \quad p_1 = \exp \{ -\lambda [\mu^* \{C\} - K(s', s' - \theta') - K(s'', \theta'' - s'')] \},$$

where  $\mu^* \{C\}$  is the expected number of disks centered at  $C$ , minus the expected number of disks centered between  $\overline{OP}_{s'}$  and  $\overline{OP}_{s''}$  having zero radius, i.e.,

$$(4.12) \quad \begin{aligned} \mu^* \{C\} &= \mu \{C\} - \frac{\lambda b}{\beta} \cdot \frac{s'' - s'}{r} \cdot \int_u^w dv \int_{v-\beta}^{v+\beta} z F(b | \frac{v-z}{\beta} |) dz \\ &= \mu \{C\} - \frac{\lambda b}{r} (s'' - s') \int_u^w dv \int_{-1}^1 (u + v\beta) F(b|u|) du. \end{aligned}$$

### 5. A Special Case And Numerical Illustration

In the present section we consider, for example, the case of a uniform distribution of sphere radii, i.e.,  $F(x) = \min \left( \frac{x}{b}, 1 \right) I_{(0, \infty)}(x)$ , where  $I_A(x)$  is the indicator function. In the present case

$$(5.1) \quad \Psi(\tau) = \frac{1}{6} - \frac{\tau^2}{2} + \frac{\tau^3}{3}, \quad 0 \leq \tau \leq 1.$$

For  $0 \leq z \leq x \leq 1$  define

$$(5.2) \quad H(x, z) = \int_x^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \int_{z/\eta}^1 \Psi(\tau) d\tau.$$

In the present case (see Dwight [1])

$$(5.3) \quad \begin{aligned} H(x, z) &= \int_x^1 \frac{\eta}{\sqrt{1-\eta^2}} \int_{z/\eta}^1 \left( \frac{1}{6} - \frac{1}{2} \tau^2 + \frac{1}{3} \tau^3 \right) d\tau \\ &= \frac{1}{12} \left\{ \sqrt{1-x^2} - z\pi + 2z \sin^{-1}(x) + 2z^3 \sqrt{1-x^2}/x \right. \\ &\quad \left. - z^4 \left( \frac{\sqrt{1-x^2}}{2x^2} + \frac{1}{2} \log \frac{1+\sqrt{1-x^2}}{x} \right) \right\}. \end{aligned}$$



Thus, according to (A.13) and (5.2) the function  $Q(v, c)$  is given in the present case by:

$$(5.4) \quad Q(v, c) = \begin{cases} 0, & v \geq y_0 \\ 2y_0 H\left(\frac{v}{y_0}, \frac{v}{y_0}\right), & y_1 \leq v < y_0 \\ y_0 \left[ H\left(\frac{v}{y_0}, \frac{v}{y_0}\right) + H\left(\frac{ds}{y_1}, \frac{y_1}{y_0}\right) \right], & 0 \leq v < y_1. \end{cases}$$

Substituting  $v = \frac{cu}{\beta}$ ,  $y_0 = (1 + c^2)^{1/2}$ ,  $y_1/y_0 = c/(1 + c^2)^{1/2}$ , we obtain for  $a = u, w$

$$(5.5) \quad Q\left(\frac{ca}{\beta}, c\right) = \begin{cases} 0, & \text{if } a \geq \beta \frac{(1 + c^2)^{1/2}}{c} \\ 2(1 + c^2)^{1/2} H\left(\frac{ca}{\beta(1 + c^2)^{1/2}}, \frac{ca}{\beta(1 + c^2)^{1/2}}\right), & \text{if } \beta < a < \frac{\beta}{c}(1 + c^2)^{1/2} \\ (1 + c^2)^{1/2} \left[ H\left(\frac{ca}{\beta(1 + c^2)^{1/2}}, \frac{ca}{\beta(1 + c^2)^{1/2}}\right) + H\left(\frac{c}{(1 + c^2)^{1/2}}, \frac{c}{(1 + c^2)^{1/2}}\right) \right], & \text{if } a < \beta. \end{cases}$$

The function  $K(s, t)$  is given by substituting (5.5) in (A.15).

The proportional total visibility measure,  $W$ , on  $\bar{C}$  is defined as the random integral

$$(5.6) \quad W = \frac{1}{L} \int_{s'}^{s''} I(s) ds,$$

where  $L = s'' - s'$  is the length of  $\bar{C}$  and  $I(s)$  is a random indicator function, assuming the value 1 if and only if  $P_s$  is visible.

The moments of  $W$  on  $\bar{C}$  are computed recursively according to the algorithm described in Yadin and Zacks [3]. In Table 5.1 we provide the first ten moments of  $W$ , for the parameters  $r^* = 1$ ,  $w^* = .6$ ,  $u^* = .4$ ,  $b = .3$ ,  $s' = -1$ ,  $s'' = 1$ . We consider Poisson random fields with intensity parameters  $\lambda = 2, 6, 10$ , and inclination angles [radians]  $\phi = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$ . The moments of the mixed-beta approximation (see [3]) are given also in Table 5.1. It should be noted that the computation of Table 5.1 on an IBM-4381 requires only

25 sec' of CPU. An evaluation of the functions  $K(s, t)$  by numerical integration raises the computation time to over 3 minutes of CPU and results in loss of accuracy.

In Table 5.2 we present the parameters of the beta-mixture approximation to the *CDF* of  $W$ , namely

$$(5.7) \quad F_W^*(w) = \begin{cases} 0, & w < 0 \\ p_0 + \frac{1 - p_0 - p_1}{B(\alpha, \beta)} \int_0^w u^{\alpha-1} (1-u)^{\beta-1} du, & 0 \leq w < 1 \\ 1, & 1 \leq w. \end{cases}$$

Using the approximation given by  $F_W^*(w)$  one can compute probabilities of  $W$  sets.

Table 5.1 Moments of Visibility and Their  
Mixed-Beta Approximations\*, For  $s' = -1$ ,  $s'' = 1$ ,  $\phi = 0$ ,  
 $\frac{\pi}{8}$ ,  $\frac{\pi}{4}$ ,  $\frac{3\pi}{8}$ , and  $\lambda = 2, 6, 10$ ,  $r = 1$ ,  $w = .6$ ,  $u = .4$ ,  $b = .3$

$\lambda$	$\phi$	1	2	3	4	5	6	7	8	9	10	$\infty$
2	0	.958	.932	.914	.902	.894	.889	.886	.886	.887	.887	.841
		.958	.932	.914	.902	.893	.886	.881	.876	.873	.870	.841
	$\frac{\pi}{8}$	.955	.927	.909	.896	.888	.882	.879	.878	.879	.882	.831
		.955	.927	.909	.896	.886	.879	.873	.868	.865	.861	.831
	$\frac{\pi}{4}$	.944	.910	.887	.871	.860	.853	.848	.846	.846	.848	.791
		.944	.910	.887	.871	.859	.850	.843	.837	.832	.828	.791
	$\frac{3\pi}{8}$	.904	.847	.809	.783	.765	.752	.743	.737	.733	.732	.678
		.904	.847	.809	.783	.764	.749	.783	.729	.721	.715	.678
	6	.878	.809	.765	.735	.713	.698	.688	.680	.676	.674	.594
		.878	.809	.765	.734	.712	.696	.683	.672	.664	.654	.594
	$\frac{\pi}{8}$	.871	.798	.751	.720	.697	.681	.670	.662	.657	.655	.573
		.871	.798	.751	.719	.696	.679	.665	.654	.646	.639	.573
	$\frac{\pi}{4}$	.842	.754	.699	.661	.635	.616	.602	.593	.586	.582	.494
		.842	.754	.699	.661	.634	.614	.598	.586	.576	.567	.494
	$\frac{3\pi}{8}$	.739	.609	.532	.482	.448	.423	.406	.392	.382	.375	.284
		.739	.609	.532	.482	.447	.422	.402	.387	.375	.365	.284
10	0	.806	.703	.640	.599	.570	.549	.534	.523	.515	.510	4.20
		.806	.703	.640	.599	.569	.547	.530	.517	.506	.497	.420
	$\frac{\pi}{8}$	.794	.687	.622	.579	.549	.527	.511	.500	.492	.486	.396
		.794	.687	.622	.578	.548	.525	.508	.494	.483	.474	.396
	$\frac{\pi}{4}$	.750	.626	.551	.503	.470	.446	.429	.416	.407	.400	.309
		.750	.626	.551	.503	.469	.444	.425	.411	.399	.389	.309
	$\frac{3\pi}{8}$	.603	.440	.352	.299	.264	.240	.223	.210	.200	.193	.123
		.603	.440	.352	.299	.263	.238	.220	.206	.195	.187	.123

\*The upper value in each cell is the moment of  $W$ . The lower value is that of the beta-mixture distribution.  $\mu_{\infty} = p_1$ .

Table 5.2 The Parameters of The Mixed-Beta Distributions,  
 For  $s' = -1$ ,  $s'' = 1$ ,  $\phi = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$  and  $\lambda = 2, 6, 10$ ,  
 $r = 1$ ,  $w = .6$ ,  $u = .4$ ,  $b = .3$

$\lambda$	$\phi$	$P_0$	$P_1$	$\alpha$	$\beta$	$\sigma$
2	0	0	.8409	3.489	1.293	.12029
	$\frac{\pi}{8}$	0	.8307	3.525	1.301	.12348
	$\frac{\pi}{4}$	0	.7905	3.595	1.324	.13578
	$\frac{3\pi}{8}$	.0001	.6576	3.474	1.354	.17217
6	0	.0007	.5954	3.132	1.336	.19392
	$\frac{\pi}{8}$	.0010	.5733	3.114	1.339	.19814
	$\frac{\pi}{4}$	.0021	.4940	2.999	1.350	.21371
	$\frac{3\pi}{8}$	.0067	.2844	2.474	1.387	.25194
10	0	.0039	.4204	2.724	1.348	.23307
	$\frac{\pi}{8}$	.0045	.3956	2.683	1.352	.23705
	$\frac{\pi}{4}$	.0068	.3088	2.492	1.373	.25082
	$\frac{3\pi}{8}$	.0159	.1230	1.872	1.483	.27510

*Appendix: Analytic Derivation of  $K(s, t)$ .*

We derive here formula (4.8) of  $K(s, t)$ . Starting from (4.5), and expressing  $G(y | z, v)$  in terms of  $F(\cdot)$ , we obtain

$$(A.1) \quad K(s, t) = \frac{b}{\beta \alpha r} \int_u^w dv \int_{v-\beta}^{v+\beta} dz \int_0^{\alpha z t} F\left(\left(y^2 + \left(\frac{z-v}{\beta}\right)^2 b^2\right)^{1/2}\right) dy,$$

in which  $\alpha = (s^2 + r^2)^{-1/2}$ .

Let  $c = \alpha \beta t / b$  then, after several simple changes of variables, we can write

$$(A.2) \quad \begin{aligned} K(s, t) &= \frac{b \beta^2 t}{c^2 r} \int_{\frac{c}{\beta} u}^{\frac{c}{\beta} w} dy \int_{-1}^1 dx \int_0^{y+cx} F(b(z^2 + x^2)^{1/2}) dz \\ &= \frac{b \beta^2 t}{c^2 r} \left[ \frac{c^2}{\beta^2} (w^2 - u^2) - J(c, \beta, u, w) \right], \end{aligned}$$

where

$$(A.3) \quad J(c, \beta, u, w) = \int_{\frac{c}{\beta} u}^{\frac{c}{\beta} w} dy \int_{-1}^1 dx \int_0^{y+cx} [1 - F(b(x^2 + z^2)^{1/2})] dz.$$

Notice that  $F(b(x^2 + z^2)^{1/2}) = 1$  for all  $(x, z)$  s.t.  $x^2 + z^2 \geq 1$ . Write

$$(A.4) \quad J(c, \beta, u, w) = J_1(c, \beta, u, w) - J_2(c, \beta, u, w),$$

where

$$(A.5) \quad J_1(c, \beta, u, w) = \int_{\frac{c}{\beta} u}^{\frac{c}{\beta} w} dy \int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} [1 - F(b(x^2 + z^2)^{1/2})] dz$$

and

$$(A.6) \quad J_2(c, \beta, u, w) = \int_{\frac{c}{\beta} u}^{\frac{c}{\beta} w} dy \int_{-1}^1 dx \int_{M(y+cx, \sqrt{1-x^2})}^{\sqrt{1-x^2}} [1 - F(b(x^2 + z^2)^{1/2})] dz,$$

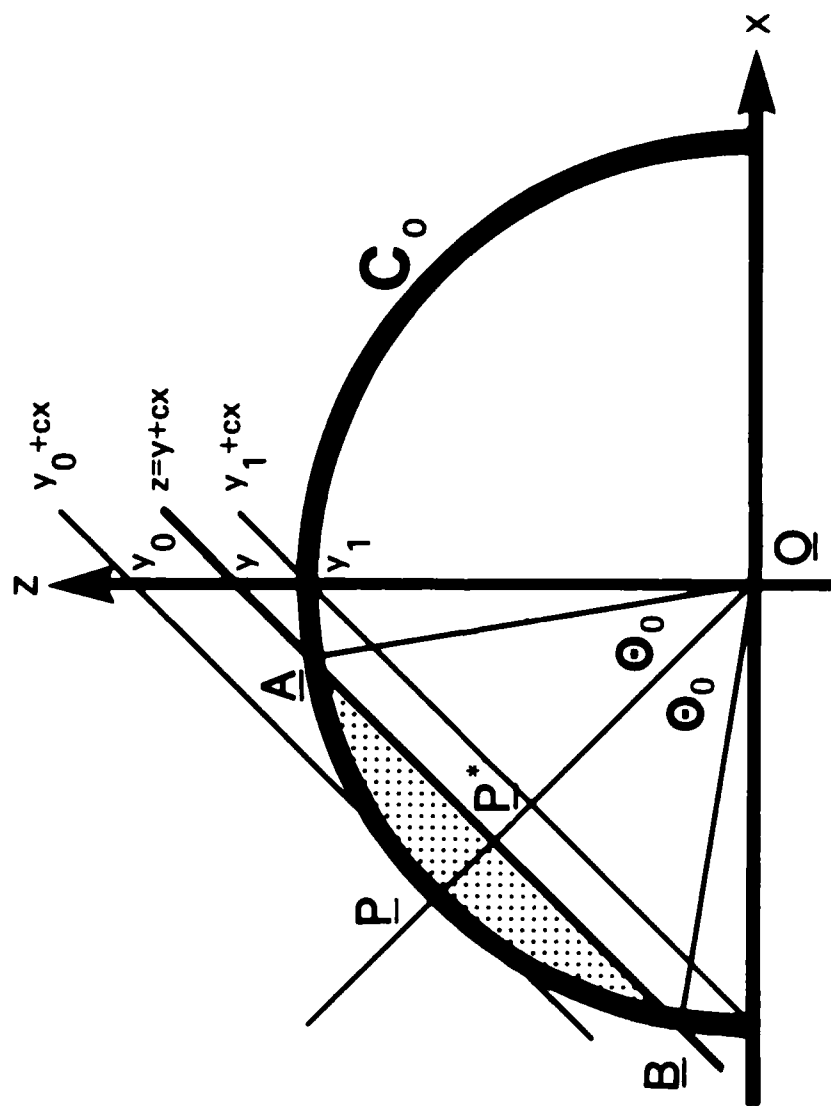


Fig. A.1. Integrated Set, Case I.

where  $M(a, b) = \min(a, b)$ .

Transforming  $(x, z)$  to the polar coordinates  $(\rho, \theta)$  we obtain

$$(A.7) \quad \begin{aligned} J_1(c, \beta, u, w) &= \int_{\frac{c}{\beta}u}^{\frac{c}{\beta}w} dy \int_0^\pi d\theta \int_0^1 \rho [1 - F(b\rho)] d\rho \\ &= \pi \frac{c}{\beta} (w - u) \Psi(0), \end{aligned}$$

where  $\Psi(t)$  is defined in (4.6).

In order to evaluate the function  $J_2(c, \beta, u, w)$ , define the function

$$(A.8) \quad Q(v, c) = \int_v^\infty dy \int_{-1}^1 dx \int_{y+cx}^\infty [1 - F(b(x^2 + z^2)^{1/2})] dz$$

Then,

$$(A.9) \quad J_2(c, \beta, u, w) = Q\left(\frac{c}{\beta}u, c\right) - Q\left(\frac{c}{\beta}w, c\right).$$

Consider the half circle  $C_0 = \{(x, z) : -1 \leq x \leq 1, 0 \leq z \leq 1, x^2 + z^2 \leq 1\}$  (see Figure A.1). Let  $\mathcal{L}$  be the line  $z = y + cx$ , having a slope  $c$ ,  $c > 0$ , and intercept  $y$ . Let  $\mathcal{L}_0$  be a line parallel to  $\mathcal{L}$ , which is tangential to  $C_0$ . Let  $y_0$  be the intercept of  $\mathcal{L}_0$ . Similarly, let  $\mathcal{L}_1$  be a line parallel to  $\mathcal{L}$ , passing through the point  $(-1, 0)$ .  $\mathcal{L}_1$  has an intercept  $y_1 = c$ . Let  $\underline{P}$  be the point at the intersection of  $C_0$  and  $\mathcal{L}_0$ . The right triangle with vertices  $\underline{Q}$ ,  $\underline{P}$ ,  $(0, y_0)$  is congruent to the triangle  $(-1, 0)$ ,  $\underline{Q}$ ,  $(0, y_1)$ . Hence  $y_0 = (1 + c^2)^{1/2}$ . It is clear that if  $v > y_0$  then  $Q(v, c) = 0$ . We distinguish two cases.

*Case I:*  $y_1 \leq v < y_0$ .

The line segment  $\overline{\underline{Q}\underline{P}}$  intersects the line  $\mathcal{L}$  at  $\underline{P}^*$ , whose distance from  $\underline{Q}$  is  $\gamma = y/y_0$  (see Figure A.1). Let  $\underline{A}$  and  $\underline{B}$  be the points at which  $\mathcal{L}$  intersects the half circle  $C_0$ . The triangle  $\Delta \underline{B}\underline{Q}\underline{A}$  is equilateral and

$$(A.10) \quad \theta_0 = \angle \underline{P}\underline{Q}\underline{A} = \cos^{-1}(\gamma).$$

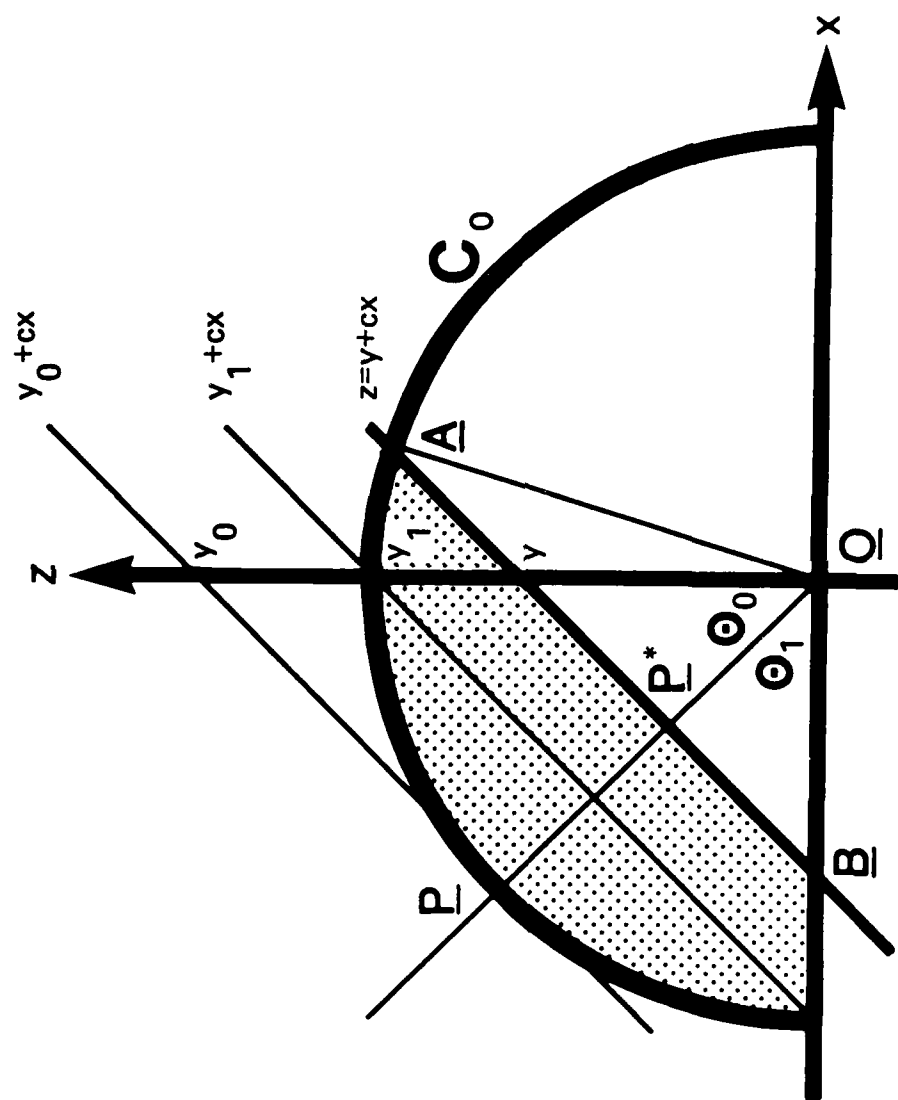


Fig. A.2. Integrated Set, Case II.



Hence, by changing to polar coordinates and making a proper rotation, we obtain

$$(A.11) \quad Q(v, c) = 2 \int_v^{y_0} dy \int_0^{\cos^{-1}(\frac{y}{y_0})} \Psi\left(\frac{y}{y_0 \cos \theta}\right) d\theta.$$

Case II:  $0 \leq v < y_1$ .

Let  $\theta_1$  denote the angle between  $\overline{OP}$  and  $\overline{OB}$  (see Figure A.2). It is immediately obtained that  $\theta_1 = \frac{\pi}{2} - \tan^{-1}(c) = \sin^{-1}\left(\frac{1}{\sqrt{1+c^2}}\right) = \tan^{-1}\left(\frac{1}{c}\right)$ . Thus, in Case II,

$$(A.12) \quad Q(v, c) = \int_v^{y_0} dy \left[ \int_0^{\theta_0} \Psi\left(\frac{y}{y_0 \cos \theta}\right) d\theta + \int_0^{\theta_1} \Psi\left(\frac{y}{y_0 \cos \theta}\right) d\theta \right].$$

Making the transformation  $\eta = \cos(\theta)$ ,  $z = y/y_0\eta$ , and changing the order of integration, we obtain

$$(A.13) \quad Q(v, c) = \begin{cases} 0, & v \geq y_0 \\ 2y_0 \int_{\frac{v}{y_0}}^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \int_{\frac{v}{y_0\eta}}^1 \Psi(z) dz, & y_1 \leq v < y_0 \\ y_0 \int_{\frac{v}{y_0}}^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \left[ \int_{\frac{v}{y_0\eta}}^1 \Psi(z) dz + \int_{M(\frac{y_1}{y_0\eta}, 1)}^1 \Psi(z) dz \right], & 0 \leq v < y_1, \end{cases}$$

Notice that in the case of  $0 < v < y_1$ , one can write

$$(A.14) \quad Q(v, c) = y_0 \left[ \int_{\frac{v}{y_0}}^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \int_{\frac{v}{y_0\eta}}^1 \Psi(z) dz + \int_{\frac{y_1}{y_0}}^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \int_{\frac{y_1}{y_0\eta}}^1 \Psi(z) dz \right].$$

Finally, according to (A.2), (A.4), (A.7) and (A.9),

$$(A.15) \quad K(s, t) = \frac{b\beta^2 t}{c^2 r} \left[ \frac{c^2}{\beta^2} (w^2 - u^2) - \frac{\pi c}{\beta} (w - u) \Psi(0) + Q\left(\frac{c}{\beta} u, c\right) - Q\left(\frac{c}{\beta} w, c\right) \right].$$

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